THE FACTORIZATION NORM AND AN INVERSE THEOREM FOR MAXCUT

Igor Balla

Based on joint work with Lianna Hambardzumyan and István Tomon

Communication complexity

Given an $m \times n$ Boolean matrix M and coordinates x, y, suppose that we have two players, Alice and Bob such that Alice receives x and Bob receives y. Alice and Bob take turns sending one bit of information to each other, until they have enough information to be able to determine $M_{x,y}$.

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$$M = \begin{array}{c} \\ x \\ \\ \end{array} \overset{\quad\quad y}{\begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}}$$

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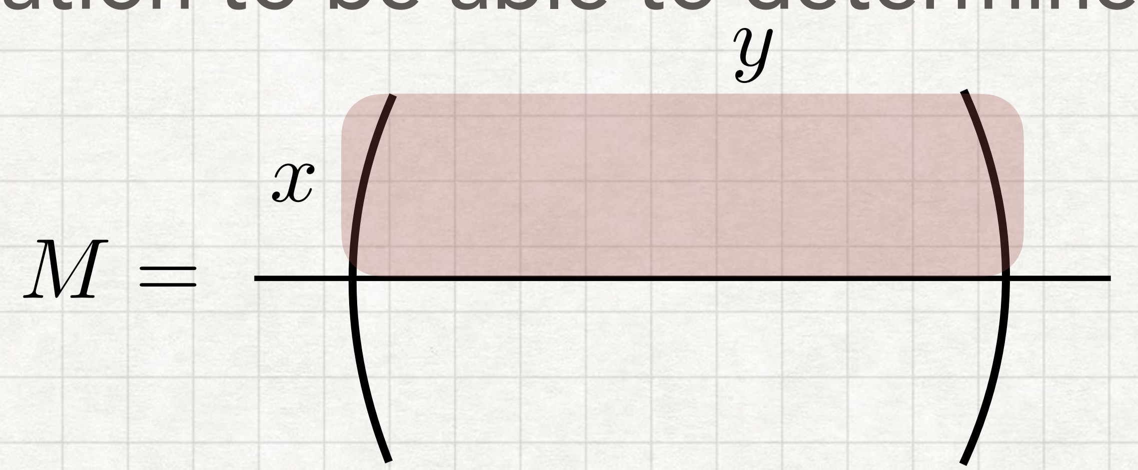

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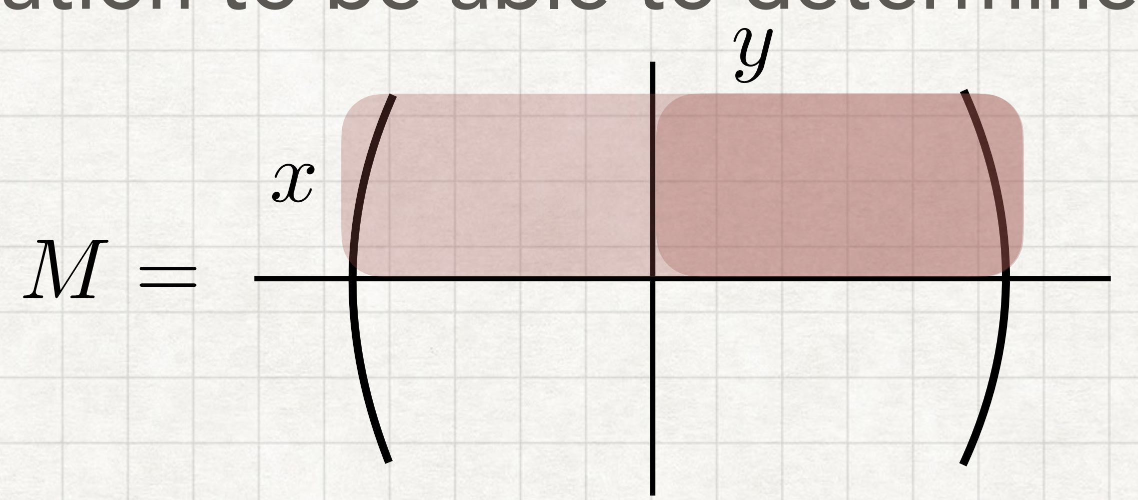

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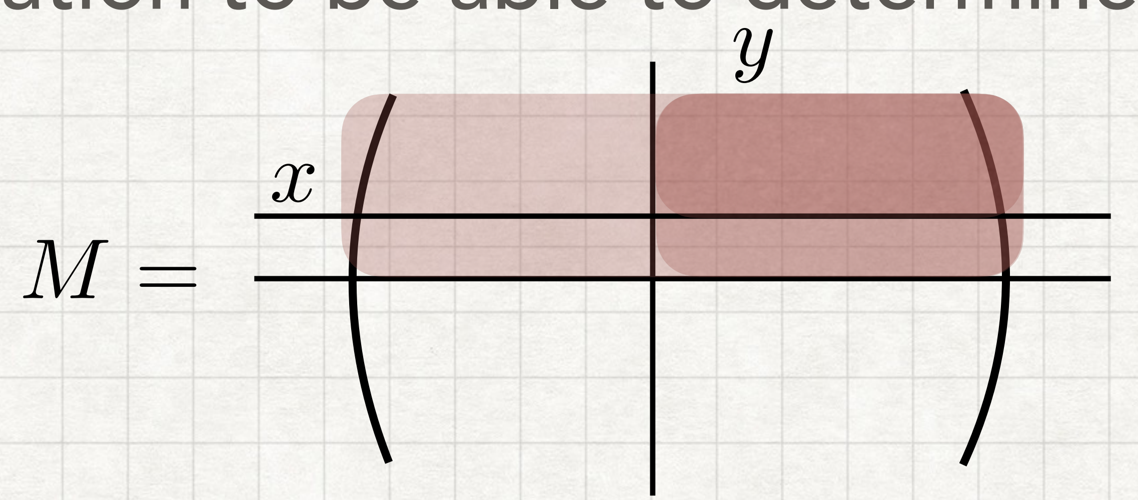

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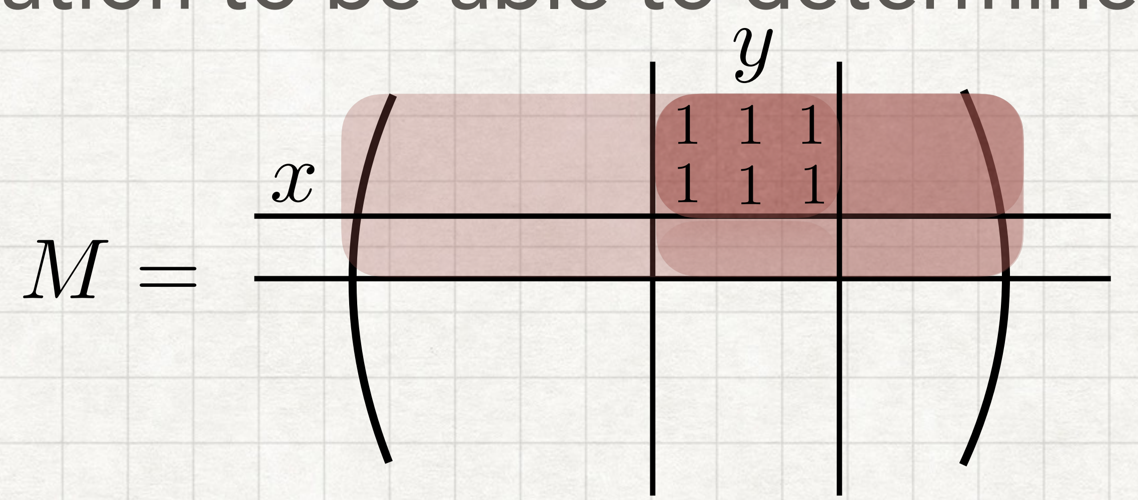

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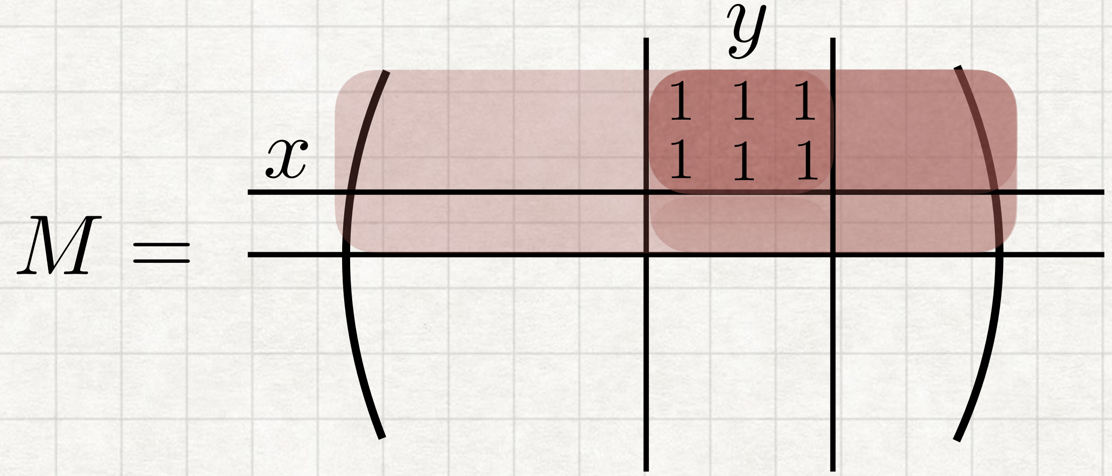

The particular choices of what bits to send in each round and what answer to output is called a **protocol** and its **length** is the maximum over all inputs x, y, of the number of rounds needed to output an answer.

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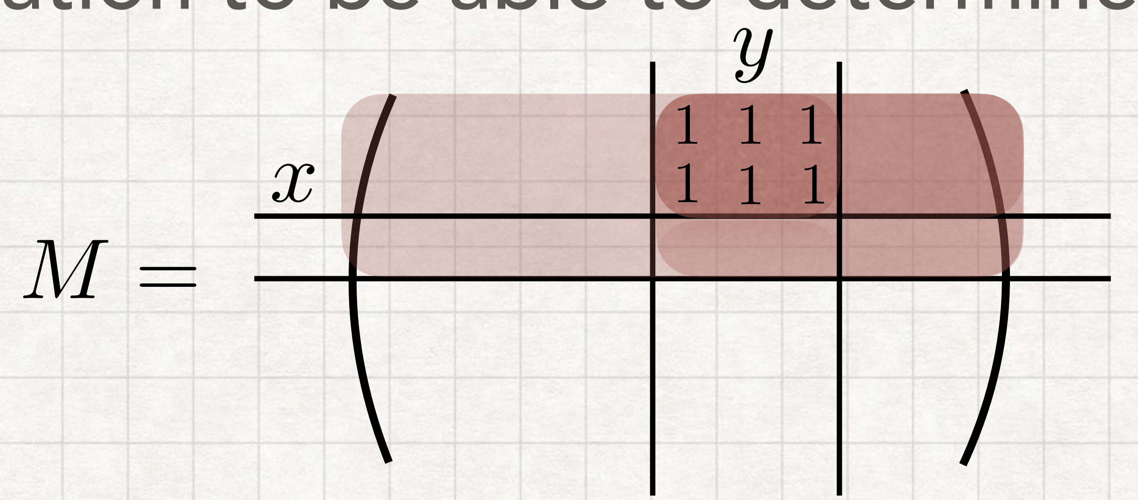

The particular choices of what bits to send in each round and what answer to output is called a **protocol** and its **length** is the maximum over all inputs x, y, of the number of rounds needed to output an answer.

The **communication complexity** $\mathrm{CC}(M)$ is the minimum length of a protocol that always successfully outputs $M_{x,y}$.

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The **randomized communication complexity** $\mathrm{RCC}(M)$ is the smallest ℓ such that there exists a probability distribution over protocols of length at most ℓ such that for any input x, y,

$$\Pr[\mathrm{output} = M(x, y)] \geq 2/3.$$

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Conjecture 1 (Hambardzumyan, Hatami, and Hatami 23): A Boolean matrix with bounded randomized communication complexity must contain a linear-sized constant submatrix.

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The **trace norm** $||M||_{\mathrm{tr}}$ is defined to be the sum of the singular values of M. The **normalized trace norm** is defined to be $\frac{||M||_{\mathrm{tr}}}{\sqrt{mn}}$.

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$$\mathrm{RCC}(M) \geq \Omega\left(\log \frac{||M||_{\mathrm{tr},1/3}}{\sqrt{mn}}\right).$$

Thus Conjecture 1 would follow if one could establish that a Boolean matrix with bounded normalized approximate trace norm must have a linear-sized constant submatrix.

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Given an $m \times n$ matrix M over the reals, the **factorization norm** γ_2 is defined as

$$\gamma_2(M) = \min_{U,V:M=UV} ||U||_{\text{row}}||V||_{\text{col}},$$

where $||U||_{\text{row}}$ denotes the maximum ℓ_2 norm of a row of U and $||V||_{\text{col}}$ denotes the maximum ℓ_2 norm of a column of V.

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Main Theorem (B., Hambardzumyan, and Tomon): Conjectures 2 and 3 hold. In particular, any $n \times n$ Boolean matrix M with $\gamma_2(M) = \gamma$ has an $n' \times n'$ constant submatrix where $n' \geq 2^{-O(\gamma^3)}n$.

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Theorem (B., Hambardzumyan, and Tomon): Let $\gamma \geq 3$ and n be sufficiently large with respect to γ. Then there exists an $n \times n$ Boolean matrix M such that $\gamma_2(M) \leq \gamma$, and M contains no $t \times t$ constant submatrix for $t \geq 2^{-\gamma+3}n$.

Theorem (B., Hambardzumyan, and Tomon): Given a Boolean matrix M such that the bipartite graph it represents has degeneracy d and is 4-cycle-free, we have $\gamma_2(M) = \Theta(\sqrt{d})$.

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Theorem (Hunter, Milojević, Sudakov, Tomon 25): For any k, there exists $c_k > 0$ such that if G is a bipartite graph of average degree d having no 4-cycle-free induced subgraph with average degree at least k, then G contains a subgraph on at most d vertices with at least $c_k d^2$ edges.

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Stronger Main Theorem (B., Hambardzumyan, and Tomon): For any γ, there exists $c_\gamma > 0$ such that if an $n \times n$ Boolean matrix M has m one-entries and $\gamma_2(M) = \gamma$, then M must contain a $t \times t$ all-ones submatrix where $t \geq c_\gamma m/n$.

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Theorem (B., Hambardzumyan, and Tomon): If a graph with m edges has MaxCut at most $m/2 + \alpha\sqrt{m}$, then it contains a clique of size $2^{-O(\alpha^9)}\sqrt{m}$.

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For matrices which can be written as $M_{x,y} = f(x-y)$, where $f: G \to \mathbb{R}$ is a function on a finite group G, conjecture 4 follows from Cohen's idempotent theorem.

Operator theory and harmonic analysis

Theorem (Goh and Hatami 25): For any Boolean matrix M having m many one-entries, there exists a blocky matrix B with at least $m/2^{2^{O(\gamma)}}$ one-entries, where $\gamma_2(M) = \gamma$, such that B is "contained" in M, i.e. $B_{x,y} \leq M_{x,y}$ for all x, y.

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Key Lemma: Let M be a nonzero $n \times n$ Boolean matrix with $p(M) \leq 1/2$ and $\gamma_2(M) \leq \gamma$. Then M contains an $n' \times n'$ submatrix M' such that $\gamma_2(M') \leq \gamma - \Omega(1/\gamma)$, $n' \geq 2^{-O(\gamma)}n$, and $p(M') \leq 1/2$.

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Proof of Main Theorem (assuming the Key Lemma): Note that it suffices to show that if $p(M) \leq 1/2$, then M has an $n' \times n'$ zero submatrix where $n' \geq 2^{-O(\gamma^3)}n$.

Indeed, if $p(M) > 1/2$, then we can instead apply the argument that will follow to $J - M$ (since $\gamma_2(J - M) \leq \gamma_2(J) + \gamma_2(M) = \gamma + 1$ and $p(J - M) < 1/2$) to get a linear-sized zero submatrix of $J - M$, which corresponds to an all-ones submatrix of M.

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Now assume $p(M) \leq 1/2$ and as long as we don't get a zero matrix, repeatedly apply the Key Lemma to get a sequence of submatrices.

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Since the dimensions of our matrix only shrink by a factor of $2^{-O(\gamma)}$ at each step, we have $n' \geq 2^{O(\gamma^3)}n$, as desired. $\square$

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Also recall that the **Frobenius** inner product of matrices $\langle A, B \rangle_{\mathsf{F}} = \sum_{i,j} A_{i,j} B_{i,j}$ satisfies $\langle xx^\intercal, yy^\intercal \rangle_{\mathsf{F}} = \langle x, y \rangle^2$.

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However, $\sum_{r=1}^{m} d_r$, $\sum_{c=1}^{n} d'_c$, and $\sum_{i=1}^{m} \sum_{j=1}^{n} M_{i,j}^2$ all count the number of ones in M, a contradiction. $\square$

Key Lemma: Let M be a nonzero $n \times n$ Boolean matrix with $p(M) \leq 1/2$ and $\gamma_2(M) \leq \gamma$. Then M contains an $n' \times n'$ submatrix M' such that $\gamma_2(M') \leq \gamma - \Omega(1/\gamma)$, $n' \geq 2^{-O(\gamma)}n$, and $p(M') \leq 1/2$.

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Repeat the following procedure. Suppose $M^{(i)} = U^{(i)}V^{(i)}$ is already defined, where the size of $M^{(i)}$ is $m_i \times n_i$. Stop if either (a) $M^{(i)}$ is an all-zero matrix, or (b) $n_i < n_0/2$, or (c) $m_i < m_0/2$.

Case 1: If there is a brilliant row r, then let $u_1, \ldots, u_{m_i}$ be the rows of $U^{(i)}$.

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Now let $M^{(i+1)} = U^{(i+1)}V^{(i+1)}$ and observe that $M^{(i+1)} = U^{(i)}V^{(i+1)}$, so $M^{(i+1)}$ is a submatrix of $M^{(i)}$.

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Moreover, $M^{(i+1)}$ is an $m_i \times (n_i - t_i)$ matrix, where t_i is the number of one entries in row r of $M^{(i)}$.

Now we compute that

$$||u_j'||^2 = ||u_j||^2 - \frac{\langle u_r, u_j \rangle}{||u_r||^2} \le ||u_j||^2 - \frac{\langle u_r, u_j \rangle}{\gamma},$$

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and thus

$$||U^{(i+1)}||_{\mathsf{F}}^2 = \sum_{j=1}^{m_i} ||u_j'||^2 \leq \sum_{j=1}^{m_i} ||u_j||^2 - \frac{1}{\gamma} \sum_{j=1}^{m_i} \langle u_r, u_j \rangle^2 \leq ||U^{(i)}||_{\mathsf{F}}^2 - \frac{t_i}{\gamma},$$

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Case 2: If there is no brilliant row, then by the previous lemma, there is a brilliant column and we proceed similarly to the previous case (but with roles of U and V reversed).

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where the last inequality follows since row r is brilliant.

Case 2: If there is no brilliant row, then by the previous lemma, there is a brilliant column and we proceed similarly to the previous case (but with roles of U and V reversed).

Note that this process must stop at some point since at each step, we are making a vector 0 by projecting. So let I be the step at which the process stopped.

Since at each step we lose at most $0.1n_0$ rows or columns and we didn't stop at step $I-1$, we must have $m_I \geq 0.4m_0$ and $n_I \geq 0.4n_0$.

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Otherwise suppose that (b) happens, i.e. $n_I < n_0/2$.

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Thus (b) can only happen if we were always in case 1 and so we have $m_I = m_0$ and $n_0 - \sum_{i=1}^{I-1} t_i = n_I < n_0/2$. Thus $\sum_{i=1}^{I-1} t_i = n_I < n_0/2$.

Therefore $$||U^{(I)}||_{\mathsf{F}}^2 \le ||U^{(0)}||_{\mathsf{F}}^2 - \sum_{i=0}^{I-1} \frac{t_i}{\gamma} \le n_0\gamma - \frac{n_0}{2\gamma}\,.$$

Therefore $$||U^{(I)}||_{\mathsf{F}}^2 \le ||U^{(0)}||_{\mathsf{F}}^2 - \sum_{i=0}^{I-1} \frac{t_i}{\gamma} \le n_0\gamma - \frac{n_0}{2\gamma}.$$

Now let U' be the submatrix of $U^{(I)}$ obtained by keeping only rows u that satisfy $||u||^2 \le \gamma - \dfrac{1}{4\gamma}$ and let m' be the number of rows of U'.

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Then $n_0\left(\gamma - \frac{1}{2\gamma}\right) \geq ||U^{(I)}||_\mathsf{F}^2 \geq (n_0 - m')\left(\gamma - \frac{1}{4\gamma}\right)$, so $m' \geq \frac{n_0}{4\gamma^2}$.

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Thus $N' = U'V^{(I)}$ is an $m' \times n_I$ submatrix of M such that

$$\gamma_2(N') \le ||U'||_{\text{row}}||V^{(I)}||_{\text{col}} \le \sqrt{\gamma - 1/(4\gamma)} \cdot \sqrt{\gamma} = \gamma - \Omega(1/\gamma)$$

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Since $n_I \ge .4n_0$, each row of N' has at most $0.1n_0 \le n_I/2$ ones and hence $p(N') \le 1/2$.

Finally, let $m = \min(m', n_I) \geq n_0/(4\gamma^2) \geq 2^{-O(\gamma)}n$ and note that on average, a random $m \times m$ submatrix N of N' will satisfy $p(N) \leq 1/2$.

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N is the desired matrix since $\gamma_2(N) \leq \gamma_2(N') \leq \gamma - \Omega(1/\gamma)$.

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The scenario in which (c) occurs can be handled in a similar way.

$\square$


thank you
danke
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ngiyabonga
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рахмат
Баярлалаа
спасибо
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go raibh maith agat
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kop khun krap
terima kasih
arigatô
ありがとう
감사합니다
tapadh leat
dank je
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barka
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kia ora
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