

# THE MINRANK OF RANDOM GRAPHS OVER ARBITRARY FIELDS

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**Theorem** (Grimmett and McDiarmid 75): For  $p$  fixed, the random graph  $G \sim G(n, p)$  has chromatic number  $\chi(G) = \Theta(n / \log n)$  with high probability.

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For finite fields  $\mathbb{F}$  with  $|\mathbb{F}| \leq n^{O(1)}$ , this result was already proven recently by Golovnev, Regev, and Weinstein.

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**Lemma:** If  $M \in \mathbb{F}^{n \times n}$  is a matrix with  $M_{i,i} \neq 0 \ \forall i$ , then

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On the other hand, any independent set in  $G$  corresponds to a full rank submatrix of  $M$ , and so must have size at most  $\text{rk}(M)$ . □

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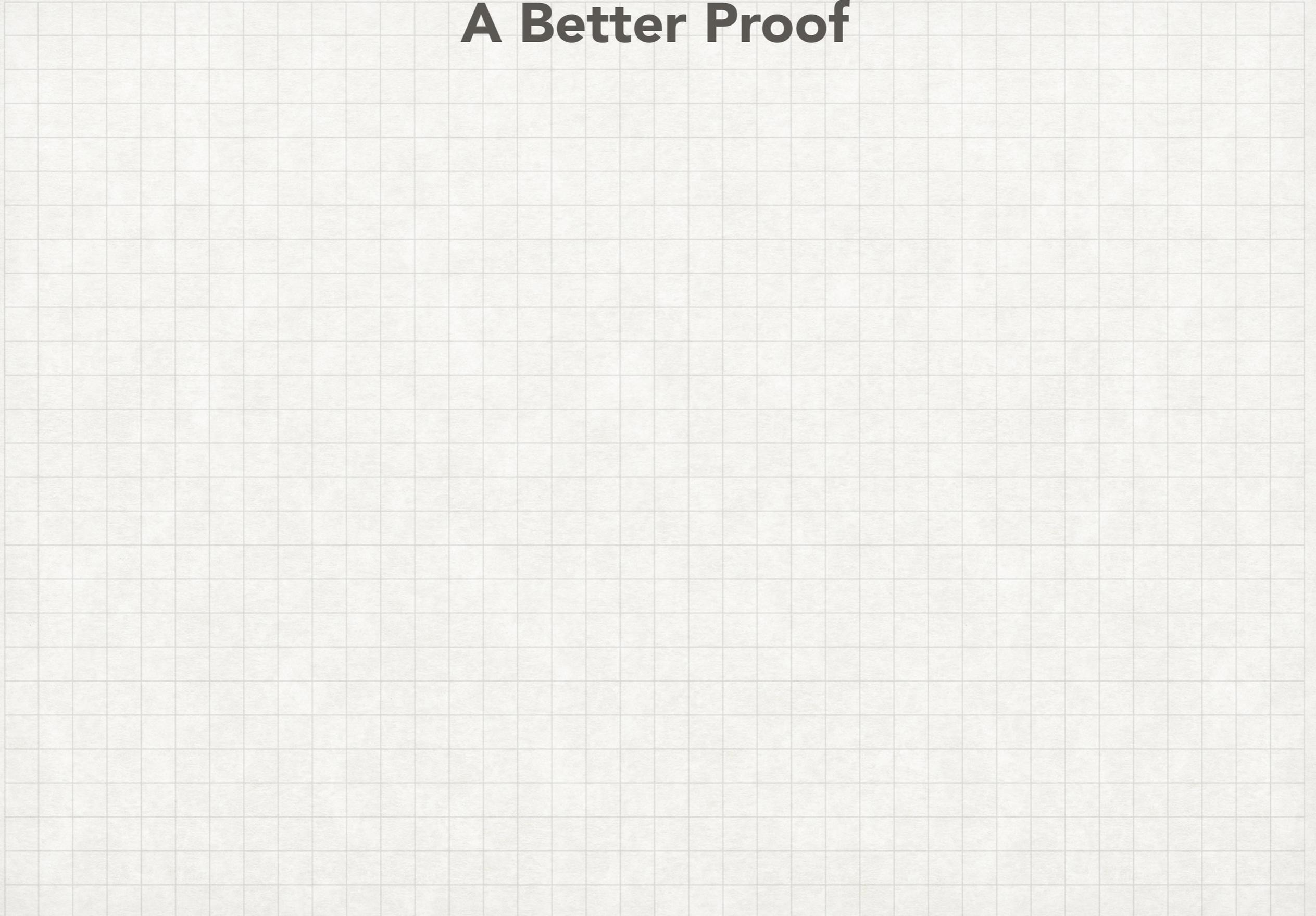
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If  $M \in \mathbb{F}^{n \times n}$  is a matrix of rank  $k$  with sparsity  $s$  which is sufficiently "nice", then each row and column will have  $\approx s/n$  nonzero entries and so the  $k$  linearly independent rows/columns determining  $M$  will each have  $\approx ks/n$  nonzero entries.

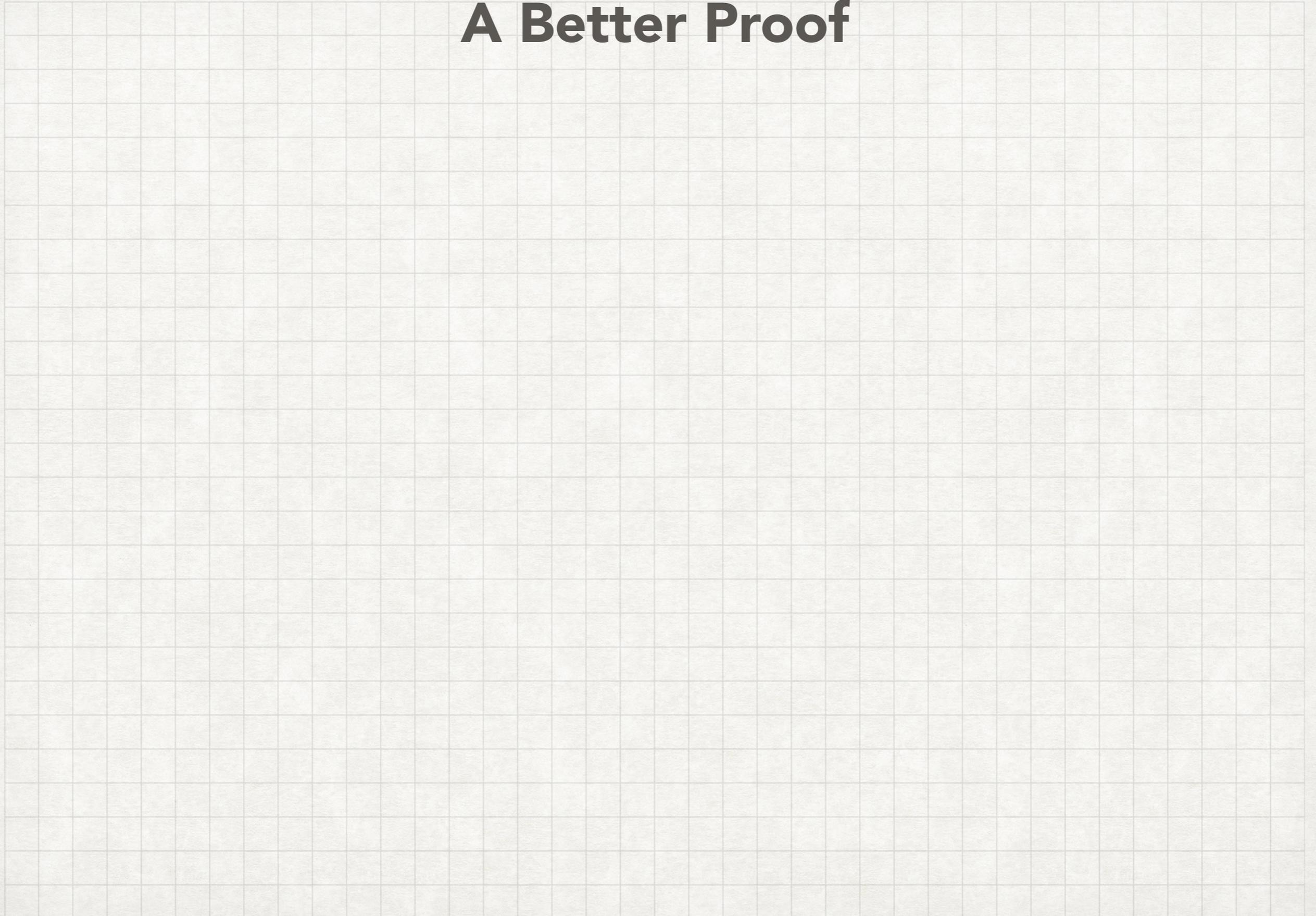
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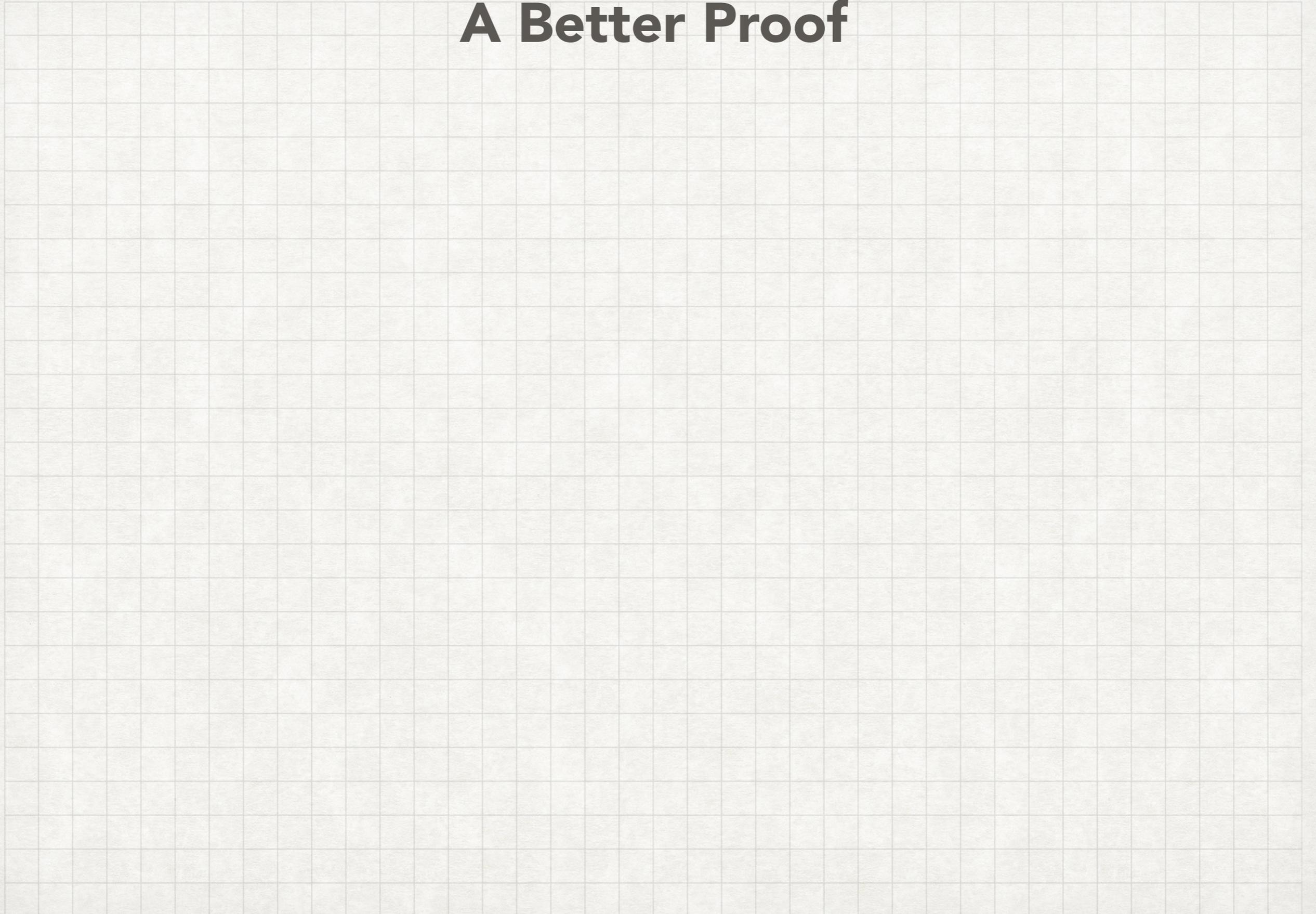
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**Lemma** (Golovnev, Regev, Weinstein 17): Every rank  $k$  matrix  $M \in \mathbb{F}^{n \times n}$  has a "nice"  $n' \times n'$  principal submatrix of rank  $k'$  such that  $k'/n' \leq k/n$ .

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**Theorem** (Rónyai, Babai, Ganapathy 01): If  $(f_1, \dots, f_m)$  is a vector of polynomials over a field  $\mathbb{F}$  with degree  $\leq d$  in  $N$  variables, then the number zero-patterns of this vector as the variables range over all possible elements is at most  $\binom{md + N}{N}$ .

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**Claim:** Each entry of  $M$  is a polynomial of degree  $\leq k + 1$  in the variables of  $E$ .

Thus we have  $n^2$  polynomials of degree  $\leq k + 1$  in  $2ks/n$  variables, so

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**Correct Claim:** Each entry of  $\det(C')M$  is a polynomial of degree  $\leq k + 1$  in the variables of  $E$ .

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**Done!**